Notes on Variables

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This document assumes some knowledge about math; as long as you have completed high school or are going through it, then you most likely recognize a lot here. I will still add context to as many symbols and concepts as I can to ensure anyone can follow through, but regardless of that, if you don't, that's also okay; take your time, search terms, seek tutorials from other channels, or if I upload one on the topic, you can try it alongside them. If you feel stuck, you can always leave it now for a bit and return later with a clearer mind. you a better insight into variables; you don't have to understand everything mentioned or know them deeply rightaway.
the main reference I used in understanding the topic is William Weiss and Cherie D'Mello's "Fundamentals of Model Theory" chapter 0, which can be found publically at math.toronto.edu/weiss/model_theory.pdf – it's not exactly an easy read, I don't personally understand it fully. I mostly studied just enough to infer this content and explain it in a simple way, you can say this content is also an attempt to understand the topic alongside you. If anything I say here is off you can tell me, these notes are an iterative process and I will update them whenever I or someone finds something off. Making these notes, or later videos, is a process I will probably take time on, and won't be fast-produced. I am working on a way to automate the process such that I would only have to write content without dealing with much formatting, so wish me luck.

Statements On Variables

You may have stumbled over variables while you were asked to find or solve for "x", And perhaps you did wonder what even is x, and why did we suddenly introduce such letters into math?

A common misconception is assuming x is just a placeholder or a constant, where it can't simply be replaced and can also vary. to understand this better lets say we have $x \in \mathbb{N}$ a Free Variable member of natural numbers.

Natural Numbers are the elements of the set containing $1, 2, 3, 4, \ldots$ when we say $\in \mathbb{N}$ we basically mean this variable is member of the set $\mathbb{N}$ which represents natural numbers.

And we have the statements:

$$\underbrace{x^2 - 5x + 6 = 0}_{1}, \quad \underbrace{x - 2 = 0}_{2}$$

about our variable x, but let's slow down, and choose only the (1) statement to be true.

which solution is standard primary or middle school math, and can be solved by factoring:

$$(x-2)(x-3) = 0$$

Now, looking closely at this statement, which is implied from the statement (1) to be also true. Allowing us to see that both can be true only when $x = 2, x = 3$. But what does that mean? Does it mean that x is the number 2 or 3?

The answer is that x can be either 2 or 3 to make that statement true. Which leads us to define what we meant by a free variable. and when we said "let there be $x \in \mathbb{N}$". it already has been a variable.

It is a mathematical object that exists within a domain we chose to be $\mathbb{N}$; a free variable is basically one that exists in this domain and can have any subset of values within it as long as it satisfies the statements that contain it in our system.

Let's explore this idea with our second statement (2).

$$x - 2 = 0 \implies x = 2$$

The symbol $\implies$ denotes "implies". When we say $S1 \implies S2$ we mean if S1 is true then S2 is implied to be also true.

We saw that statement (1) implies that x can take two values, but (2) says x can only be 2 and not any other element for it to be true. Basically, we are sort of constraining our value with each statement:

- First, we defined it within a domain of natural numbers. Meaning if it had a negative value, it would have been contradictory.
- Secondly, our statement (1) chooses only a subset of our domain that can make it true; if we choose perhaps 6, it would have been contradicting our statement.
- Lastly, our statement (2) chooses a subset that doesn't contradict our first statement or the definition. making x can only be 2 in this system of statements.

Seeing that our system of Statements (1), (2) is consistent and has a value within our domain. Let's see what happens when we introduce a statement:

$$\underbrace{x^2 - 5x + 4 = 0}_{3} \implies (x-4)(x-1) = 0$$

Which is similar to statement one; by the method of solving, we see that it has 2 values too. Either $x = 4$ or $x = 1$.

But in our established system, statement (1) states that x can only take 3 or 2, and now by introducing statement (3), we find that the statement is inconsistent with (1) and (2). making it contradictory if (1), (2) must be true.

And if the three statements must be true, then the system itself is contradictory/inconsistent, and there is no such solution for x in its domain.

Seeing that x in our previous system can hold a value or two. How many can x hold?

$\mathbb{Z}$ is the set of integers whose elements are: $0, \pm 1, \pm 2, \ldots$

Which is a funny question because it can hold as much as its whole set of definition, for example:

$$x + x = 2x$$

The statement is true for all x, x Here can be anything, and the statement will be true, which we sometimes call an identity statement, and you may have seen many forms of it. Such can include Trigonometric function identities, which are true for any value of those functions in their stated domain. as example:

$$sin^2(x) + cos^2(x) = 1$$

Identity statements are similar to what we call a tautology, even if they aren't the same; in language, it means a statement that's circular, as if what we build on loops back to where we began and vise versa.

Although in mathematical logic it's a statement that's always true weither if the statements used inside it as premises are true or false: basically the premises grant the conclusion logically.

if we have a statement about variable x called P(x) or called "the statement" here, and it can be true or false. an example of a tautology would be "The negation of the negation of the statement is the statement."

Where an Identity can depend on more structure than of that purely logical, such structures can be operators (e.g. "$+$", "$-$", "$\times$"), functions, and etc. making it broader than a tautology but not neccesserily one.

But aside of Identities, the last example we will explore this chapter; Another system that can have more than 1 solution: a system of inequalities. Let's say we have a free variable $x \in \mathbb{Z}$ and have the following statements:

$\mathbb{R}$ is the set of real numbers which includes any rational (numbers of the form $\frac{a}{b}; \quad a, b \in \mathbb{Z}$), or irrational numbers ($\sqrt{2}, \pi, \ldots$).

$$\underbrace{x \leq 10}_{*}, \quad \underbrace{x > -2}_{**}$$

using statement $(*)$ we can see that any value in $\mathbb{Z}$ less than or equal to 10 makes it true. And checking statement $(**)$ narrows it down to where x must be greater than -2. This system is consistent and does give us a wider set of values: $-1, 0, 1, \ldots, 10$. And now if we redefined x to be a member of the real numbers, we get a really infinitely wide, even if bounded range in our system then our solution would be within $(-2, 10]$.

Now that we have viewed variables so far in this light, we can define that a constant is also a mathematical object that does exist within a domain but shouldn't range over more than one value; for example, the number "2" can't have more than one value. Same applies to $\pi, e, \sqrt{2}, \ldots$.

The special thing about constants is that they must have a single value and can't vary, meaning if we have the statement:

$$(c-4)(c-5) = 0$$

about a constant $c \in \mathbb{N}$, then it's underdetermined; it's either 4 or 5, but only one is true. so to know which a new statement should be introduced.

To notice how that is different from variables: if c was instead a variable, then either $c = 4$ and $c = 5$ make the statement true; here, the constant is either of these values, and only one of them is true.

To describe ranges, we often use parentheses: (a, b) meaning that any value between a and b but not equal to either.
And we can also use brackets: $[a, b]$ for any value of a or b or in between. Or for something hybrid, we have our example.

We saw so far what differs a variable from a constant and how a variable can change, which shows its importance.
For those who may ask why we introduce letters to math as variables: It's for generalization. So, as an Example: Let's take a specific case for any statement, Let's say:

$$(x-2)(x+2) = x^2 - 2x + 2x - 4 = x^2 - 2^2$$

That's fine on its own, but when we also look at:

$$(x-3)(x+3) = x^2 - 3^2,\ (x-4)(x+4) = x^2 - 4^2, \ldots$$

We notice a pattern; let's have a variable $y \in \mathbb{R}$ and plug it in instead of the numbers to see if it. holds the statement true:

$$(x-y)(x+y) = x^2 + xy - xy - y^2 = x^2 - y^2$$

meaning that the statement was actually more general, and this is also another example of an identity statement you may have encountered.

And as seen, we use variables that, for a statement, can have as many values as its whole domain to have a more general statement instead of one about a specific value or case. Hence why, whengeneralizing, we always introduce a new variable and prove its consistency and existence. leading to the thinking "What value of this statement can be generalized into a wider pattern, and what possibly can be its domain?"

Of course, generalization isn't the only use of variables; as we saw from the examples, it's used to solve for cases where a statement or a system is true. same thinking as "what value can x possibly have to make that statement true?", and later we will see the last use case here, which is constructing mathematical structures.

Placeholders In Mathematical Structures

The symbol "$\rightarrow$" in the context of defining a function or relation is just to indicate which domain is mapped to which image.

In the last examples, we seemingly only dealt with free variables, but in reality there are two types: a free variable and a bound variable. We left out this insight to see how a variable can vary or hold multiple values, but there was the other aspect of the argument: is a variable always a placeholder?

And to answer that, let's understand what it means to have a placeholder: the variable must be swapable within its structure without changing it. I know it sounds kind of vague; what do you mean by swapable?

A function in itself can be seen as a mirror for each element in the domain it's defined on, giving it only a singular image.

It's quite simple; to demonstrate, let's look at something you may come across in secondary or primary school: functions.
lets define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = x^2$$

Would it change anything to have the function defined as $f(t) = t^2$? You can plug in whatever value you want in both functions; it behaves the exact same and doesn't change anything except for the label in its definition.

The summation operator:

$$\sum_{i=0}^{n} f(i) = f(0) + \cdots + f(n)$$

On the other hand, integrals can be seen as summing the infinitely many small area chunks of a function, giving us the total signed area under the curve of that function

Another example might be the sum operator:

$$\sum_{i=0}^{n}(i^2) = \sum_{r=0}^{n}(r^2)$$

We can see clearly how interchangeable "i" was inside the sum operator; it doesn't change much in the argument or the statement except the label, and the same goes for integrals:

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

You may think about how functions, relations (e.g. relation of a circle), etc. are different from having a statement like those from the examples in the previous sections.

A way to look at these is that they are maps: they have both the "inputs" & "outputs", the destination and starting points. When we define a function, we define a set of pointers from where each starting point leads to which destination, and the variables we use to define the

function are bound to it by that definition.

Using the map analogy: the map itself stays pointing to the same points no matter the name you give them, and "x" or "t" are generic names, just as we did with "inputs" and "starting points" in our analogy; they are interchangeable. same with any structure that is defined with a set that bounds a variable.

logarithms i.e. $x = log_b(a)$ can be defined as the inverse of exponants i.e. $b^x = a$ which helps us find for a base b which power x it must be raised to, such that it gives us the result a

You may also think what's the purpose of such distinction, and when a variable is a placeholder what does that allow us? To demonstrate, let's have the statement:

$$f(e^x) = 1 - x$$

and we need $f(x)$ to perhaps find the solutions of a system or find its domain or range, if we let:

$$t = e^x \implies x = ln(t)$$

we will also find $f(x)$, as demonstrated below:

$$f(t) = 1 - ln(t) \implies f(x) = 1 - ln(x)$$

showing us that f maps $(0, \infty)$ to the Real numbers which could have also been found by knowing the range of "e^x".

We can't simply do the same with free variables, and to be more precise, what bound variables here allowed wasn't the substitution but the renaming of the variable from "x" to "t" within the definition of the function.

the symbol "∞" denotes infinity, it doesn't exist in the real numbers we just defined unless we extend it to include it, but regardless if we view real numbers as a line it's the unreachable bound that every real number must be lesser than, and we don't include it because it's not a member of reals.

same applies to indefinite integrals, ones with boundaries, which allows us to have techniques of solving them like u-substitution.

So what's a bound variable? It's basically a variable whose definition is bound to a structure or proof. Examples of proofs would be in mathematical logic, which is a nice subject to get an overview of its basics, and I do recommend giving it a try.

Funny enough, we do have to know some of its basics to actually understand the difference between bound and free variables better and more rigorously. The next section is optional but recommended.

Quantifiers – Free vs Bound

Recommend a glimpse into what's model theory if you want a deeper understanding of the topic it will help demystify some aspects of math. but don't force yourself through it, return to the title page, and you can check: youtube.com/watch?v=xNJHw8E_36g which aims to simplify it

In logic, we have different operators to build or prove our arguments. Two of them are called quantifiers, lets say we have a statement about a variable x denoted as $P(x)$:

- to say that for every and each x the statement $P(x)$ is true, we instead use an upside down A standing for "for all x": $\forall x, P(x)$
- and to say that there exist at least one x such that the statement $P(x)$ is true, we instead use an inverted E standing for "there exist an x": $\exists x, P(x)$

Those two quantifiers are present in almost everything you may encounter, in the definitions of functions, relations, limits, etc. Which is actually the sole differentiator between a Free and a Bound Variable:

A Free Variable is basically one defined only by its domain; it can have a wide range of outcomes within its domain based on the statements that constrain and narrow it down. It's not interchangeable like bound variables, and it's mostly used in finding the truth about a statement or proof.

A Bound Variable is different, as it's defined by its domain and a quantifier bounding it to a structure. it can be interchanged, as I will explore why soon, and it's used in constructing mathematical structures/objects or proofs.

$S1 \iff S2$: is simply stating that if one is true or false, the other should be

and the reason why it's interchangeable can be found by this beautiful tautology:

$$\forall x, \quad P(x) \iff \forall y, \quad P(y)$$

Meaning that it's for a statement $P(x)$ about a variable x, true for each x, to imply that if the same statement was about a variable y, then it's true for each y. In simplier terms: doesn't matter if it's for all x or for all y as long as the statement that follows is about the variables specified and the statement stays the same, and since the examples we provided before are defined through these quantifiers, they follow suit.

And on ending notes: To understand variables is to better differentiate between constructing and proving in mathematics. To think more rigorously about what can be overlooked. But most importantly, to stay curious and be more confident in following that curiosity.